Ch3. Fundamental Properties of Stochastic Impulsive System w/ Time Delay

Recall

$$(3.1)\quad \begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \neq \tau_K(x(t^-)) \\ \Delta x := x(t) - x(t^-) = I(t, x_{t^-}), & t = \tau_K(x(t^-)) \end{cases}$$

where
- $x \in \mathbb{R}^n$: system state random process.
- $f \in \mathbb{R}^n$, $g \in \mathbb{R}^{n\times m}$
- $\tau_K \in C^2(\mathbb{R}^n, \mathbb{R}_+)$ an impulsive hypersurface $(K \in \mathbb{N})$
  s.t. $0 = \tau_0(x) < \tau_1(x) < \cdots$ and $\lim_{K\to\infty} \tau_K(x) = \infty$ for $\forall x \in \mathbb{R}^n$

< Assumptions >

① for the solution $x(t)$, $x(t^+) = x(t)$ (right conti.)

② the impulsive amount $I(\cdot)$ is $\mathcal{F}_{\tau_K}$-adapted.

③ the initial condition $x_{t_0} = \phi(s)$, $s \in [-h, 0]$
  where $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-h,0]\times\Omega ; \mathbb{R}^n)$

Then, $$x(t) = \phi(0) + \underbrace{\int_{t_0}^{t} f(s, x_s)ds}_{\text{Riemann integral (a.s.)}} + \overbrace{\int_{t_0}^{t} g(s, x_s)dW(s)}^{\text{Itô integral}}$$

and $\mathbb{E}[\int_{t_0}^{t} g(s, x_s)dW(s)] = 0$, $\mathbb{E}\|\int_{t_0}^{t} g(s,x_s)dW(s)\|^2 = \int_{t_0}^{t}\mathbb{E}\|g(s,x_s)\|^2 ds$.

Def 3.1 two random process $x(t,\omega)$, $y(t,\omega)$ are indistinguishable if $\mathbb{P}\{\omega \mid x(t,\omega) = y(t,\omega) \text{ for all } t \geq 0\} = 1$, we say $x = y$ (a.s.)

Def 3.2 Let x, y be solutions of (3.1) defined on the intervals J_1, J_2, resp. w/ $J_1 \subset J_2$
where J_1, J_2 have the same closed left endpoints.
- If $x(t) = y(t)$ (a.s) $\forall t \in J_1$, then y is said to be a proper forward continuation of x or continuation of x.
  In this case, x is said to be continuable
- otherwise, x is said to be noncontinuable and
  J_1 is called the maximal interval of existence of x.

Lemm 3.1 Let $\mathbb{N}$ be the set of natural numbers, $D \subset \mathbb{R}^n$, $a, b \in \mathbb{R}_+$ w/ $a<b$, and c, e are some positive constanst. Then, the set

$$Q' = \{ x^{(n)} \in C([a,b]; D) \mid \mathbb{E}[\|x^{(n)}(t)\|^2] \le c \ \ \forall t \in [a,b] \text{ and } \mathbb{E}[\|x^{(n)}(t_1) - x^{(n)}(t_2)\|^2] \le e, \ \forall n \in \mathbb{N}, \forall t_1, t_2 \in [a,b] \}.$$

is totally D-bounded subset of $C([a,b]; D)$.

pf) Let $x^{(n)} \in Q'$ and $\varepsilon > 0$ be given.

$\frac{c}{h_1(\varepsilon)^2} < \frac{\varepsilon}{2}$ $\qquad$ $\frac{e}{h_2(\varepsilon)^2} < \frac{\varepsilon}{2}$

For this $\varepsilon > 0$, $\exists h_1(\varepsilon), h_2(\varepsilon) > 0$ s.t. $0 < \sqrt{\frac{2c}{\varepsilon}} < h_1(\varepsilon)$, $0 < \sqrt{\frac{2e}{\varepsilon}} < h_2(\varepsilon)$ $\cdots (*)$

Since $\mathbb{E}[\|x^{(n)}(t)\|^2] \le c$ and $\mathbb{E}[\|x^{(n)}(t_1) - x^{(n)}(t_2)\|^2] \le e$,

by Tchebychev's Ineq. (T.I), we have :

$$\begin{cases} \mathbb{P}\{\omega \in \Omega \mid \|x^{(n)}(t)\| > h_1(\varepsilon)\} \overset{\text{T.I}}{\le} \mathbb{E}[\|x^{(n)}(t)\|^2] / h_1(\varepsilon)^2 \le \frac{c}{h_1(\varepsilon)^2} \overset{(*)}{<} \frac{\varepsilon}{2} \\ \mathbb{P}\{\omega \in \Omega \mid \|x^{(n)}(t_1) - x^{(n)}(t_2)\| > h_2(\varepsilon)\} \\ \qquad \overset{\text{T.I}}{\le} \mathbb{E}[\|x^{(n)}(t_1) - x^{(n)}(t_2)\|^2] / h_2(\varepsilon)^2 \le \frac{e}{h_2(\varepsilon)^2} \overset{(*)}{<} \frac{\varepsilon}{2} \end{cases}$$

Hence, $\mathbb{P}\{\omega \in \Omega \mid \|x^{(n)}(t)\| > h_1(\varepsilon) \text{ or } \|x^{(n)}(t_1) - x^{(n)}(t_2)\| > h_2(\varepsilon)\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \mathbb{P}\{\omega \in \Omega \mid \|x^{(n)}(t)\| \le h_1(\varepsilon) \text{ and } \|x^{(n)}(t_1) - x^{(n)}(t_2)\| \le h_2(\varepsilon)\} > 1 - \varepsilon$

By Arzela-Ascoli's thm, K_ε : compact and by Prokhorov's thm,

Q' : totally D-bounded.

Remark 3.1 Q' is a collection of sequences which are both uniformly bounded and equicontinuous.

Recall (Tchebychev's Ineq.)

If $x : \Omega \to \mathbb{R}^n$ is a random variable s.t. $\mathbb{E}[\|x\|^p] < \infty$ for some p, then $\mathbb{P}\{\omega \in \Omega \mid \|x\| \ge K\} \le \mathbb{E}[\|x\|^p] / K^p$ for any $K > 0$.

§3.1 Existence of Solution.

Thm 3.1 Let $J \subset \mathbb{R}_+$, $D \subset \mathbb{R}^n$: open w/ $\phi(0) \in D$, $\alpha > 0$ s.t. $[t_0, t_0+\alpha] \subset J$.

(H1) for each $t \in [t_0, t_0+\alpha]$,

$f(t,\psi) \in \mathcal{L}_{ad}(\Omega : L[t_0, t_0+\alpha])$, $g(t,\psi) \in \mathcal{L}_{ad}(\Omega : L^2[t_0, t_0+\alpha])$

are continuous ftns of ψ on $\mathcal{L}^2_{\mathcal{F}_t}([-r,0] \times \Omega ; \mathbb{R}^n)$

(H2) for each $0 < \beta \le \alpha$ and for each compact set $F \subset D$,

$\exists$ a random ftn $m(t)$ s.t.

- $\|f(t,\psi)\|^2 \vee \|g(t,\psi)\|^2 \le m(t)$ (a.s.) for all $(t,\psi) \in [t_0, t_0+\beta] \times F$
- $\int_{t_0}^{t} m(s)\,ds < \infty$ (a.s.).

(H3) for each $k \in \mathbb{N}$, $\tau_k \in C^2(D, \mathbb{R}_+)$ and

whenever $t^* = \tau_k(x^*)$ for $(t^*, x^*) \in J \times D$, $\exists\, \delta > 0$ s.t.

- $[t^*, t^*+\delta] \subset J$
- $\mathbb{E}[\mathcal{L}\tau_k(x(t))] \neq 1$, for $\forall\, t \in (t^*, t^*+\delta]$ and

for $\forall\, \mathcal{F}_t$-adapted $x \in PC([t^*-r, t^*+\delta] ; D) \cap C((t^*, t^*+\delta] ; D)$

which satisfy ① $x(t^*) = x^*$

② $\mathbb{E}[\|x(s) - x^*\|^2] < \lambda$ for all $s \in [t^*, t^*+\delta]$

- If (H1) and (H2) hold, then

for almost $w \in \Omega$ and each $(t,\phi) \in J \times \mathcal{L}^2_{\mathcal{F}_0}([-r,0] ; \mathbb{R}^n)$,

$\exists$ a (local) $\mathcal{F}_t$-adapted solution $x(t) = x(t ; t_0, \phi)$ of (3.1)

on $[t_0 - r, t_0+\beta]$

- If (H1), (H2), and (H3) hold, then

the solution x leaves the hypersurface $\tau_k(x)$ in mean

i.e. $\exists\, \beta > 0$ and $\exists$ solution $x \in [t_0 - r, t_0+\beta]$ s.t. x will not intersect any impulse hypersurface at any time $t \in (t_0, t_0+\beta]$.

§3.2 Forward Continuation.

Thm 3.2

If ① the functionals f and g satisfy the conditions in thm 3.1

② $\tau_K \in C^2(D, \mathbb{R}_+)$ for some $K \in \mathbb{N}$ and $\lim_{K\to\infty} \tau_K(x) = \infty$ uniformly in x.

③ $\mathbb{E}[\mathcal{L}\,\tau_K(\psi(0))] < 1$ for $\forall (t,\psi) \in J \times PC([-r,0]:D)$ and $\forall K \in \mathbb{N}$.

④ $\forall \psi \in PC([-r,0]:D)$ for which $\psi(0^-) = \psi(0)$ (a.s.) and $\forall K \in \mathbb{N}$,

(3.21) $\quad \psi(0) + I(\tau_K(\psi(0)), \psi) \in D$

(3.22) $\quad \tau_K(\psi(0) + I(\tau_K(\psi(0)), \psi)) \le \tau_K(\psi(0))$ (a.s.)

Then, ① for each continuable solution x of (3.1), $\exists$ a continuation y of x that is noncontinuable.

② any solution x of (3.1) can intersect each impulse hypersurface at most once.

Thm 3.3 Let x be a solution of (3.1) that is defined for $\forall t \in [t_0 - r, t_0+\beta)$ where $0<\beta<\infty$, $[t_0, t_0+\beta] \subset J$.

- If x is noncontinuable, for each compact set $F \subset D$, $\exists$ a seq. $\{s_k\}_{k=1}^{\infty}$ s.t. $\begin{cases} t_0 < s_1 < s_2 < \cdots < s_k < \cdots < t_0+\beta \text{ and } \lim_{k\to\infty} s_k = t_0+\beta \quad (*) \\ x(s_k) \notin F \text{ for any } k \in \mathbb{N}. \end{cases}$

pf) Assume that $\exists$ compact set $F_1 \subset D$ s.t. $\nexists$ a seq $\{s_k\}_{k=1}^{\infty}$ which satisfies (*).
i.e. For every seq. $\{s_k\}_{k=1}^{\infty}$ which satisfies (*), $\exists k_0 \in \mathbb{N}$ s.t. $x(s_{k_0}) \in F_1$.
<claim : $\exists \beta_1 > 0$ s.t. $x(t) \in F_1$ for $\forall t \in [t_0+\beta_1, t_0+\beta)$ >

pf of claim) If for $\forall c \in (0,\beta)$, $\exists t_c \in [t_0+c, t_0+\beta)$ s.t. $x(t_c) \notin F_1$,
then we can define $c_1 = \frac{\beta}{2}$, $c_{i+1} = \max\{t_{c_i} - t_0, t_0+\beta - \frac{1}{i+1}\}$ $(i \in \mathbb{N})$.
so that the seq. $\{t_{c_k}\}_{k=1}^{\infty}$ satisfies
① $t_0 < t_{c_1} < \cdots < t_{c_k} < \cdots < t_0+\beta$
② $\lim_{k\to\infty} t_{c_k} = t_0+\beta$
③ for $\forall k \in \mathbb{N}$, $x(t_{c_k}) \notin F_1$. ↯

Let $F_2 := \overline{x([t_0-r, t_0+\beta_1])}$.
Then $F := F_1 \cup F_2$ $(\subset D)$: compact and $x(t) \in F$ for $\forall t \in [t_0-r, t_0+\beta)$.
Now, for any $t > \bar{t} \in [t_0+\beta_1, t_0+\beta)$, we have from (2.53)

$$\|x(t) - x(\bar{t})\| = \Big\| \Big(\phi(0) + \int_{t_0}^{t} f(s,x_s)ds + \int_{t_0}^{t} g(s,x_s)dW(s) + \sum_{\{k: \tau_k \in (t_0,t)\}} I(\tau_k, x_{\tau_k^-})\Big) - \Big(\phi(0) + \int_{t_0}^{\bar{t}} f(s,x_s)ds + \int_{t_0}^{\bar{t}} g(s,x_s)dW(s) + \sum_{\{k: \tau_k \in (t_0,\bar{t})\}} I(\tau_k, x_{\tau_k^-})\Big)\Big\|$$

$$\le \Big\|\int_{\bar{t}}^{t} f(s,x_s)ds\Big\| + \Big\|\int_{\bar{t}}^{t} g(s,x_s)dW(s)\Big\|$$

WLOG, assume that β_1 is sufficiently close to β so that $\tau_k \notin [t_0+\beta_1, t_0+\beta)$, $\forall k$.

Hence,

$$E[\|x(t) - x(\bar{t})\|^2] \le E\Big[\Big(\Big\|\int_{\bar{t}}^{t} f(s,x_s)ds\Big\| + \Big\|\int_{\bar{t}}^{t} g(s,x_s)dW(s)\Big\|\Big)^2\Big]$$

$(a+b)^2 \le 2(a^2+b^2)$ if $a,b \in \mathbb{R}$.

$$\le 2\cdot\Big\{E\Big\|\int_{\bar{t}}^{t} f(s,x_s)ds\Big\|^2 + E\Big\|\int_{\bar{t}}^{t} g(s,x_s)dW(s)\Big\|^2\Big\}$$

Cauchy-Schwartz Ineq. $\|$ ito isometry.

$$\le 2\cdot\Big\{|t-\bar{t}|\int_{\bar{t}}^{t} E\|f(s,x_s)\|^2 ds + \int_{\bar{t}}^{t} E\|g(s,x_s)\|^2 ds\Big\}$$

$$\overset{\circledast}{\le} 2\cdot\Big\{|t-\bar{t}|\int_{\bar{t}}^{t} m(s)ds + \int_{\bar{t}}^{t} m(s)ds\Big\}$$

$$= 2(\beta+1)\cdot|M(t) - M(\bar{t})|$$

$\Big\|\int_{\bar{t}}^{t} f(s,x_s)ds\Big\|^2$
$\le \Big(\int_{\bar{t}}^{t} \|f(s,x_s)\| \times 1\, ds\Big)^2$
$\overset{CS}{\le} \int_{\bar{t}}^{t} \|f(s,x_s)\|^2 ds \times \int_{\bar{t}}^{t} 1^2 ds$
$= (t-\bar{t})\int_{\bar{t}}^{t} \|f(s,x_s)\|^2 ds$
if $t-\bar{t} \ge 0$.

⊛ In the proof of thm 3.1, we showed that

if ① $f \in \mathcal{L}_{ad}(\Omega; L[t_0, t_0+\alpha])$, $g \in \mathcal{L}_{ad}(\Omega; L^2[t_0, t_0+\alpha])$ $(\alpha > 0)$.

② $\exists$ a (random) ftn $m(t)$ s.t. $\exists\, 0 < \beta \le \alpha$ and $\exists$ cpt set s.t.

$\forall (t, \psi) \in [t_0, t_0+\beta] \times F$, $\|f(t,\psi)\|^2 \vee \|g(t,\psi)\|^2 \le m(t)$ (a.s.) (**)

where $\int_{t_0}^{t} m(s)\,ds < \infty$ (a.s.).

Then, $M(t) := \int_{t_0}^{t} m(s)\,ds$ $(t \in [t_0, t_0+\alpha])$ is absolutely continuous and $M(t)$ is bounded (a.s.) $(\because m \in L^1([t_0, t_0+\alpha], \mathbb{R}^+))$

Since $F = F_1 \cup F_2$: compact & $x(t) \in F$ for $\forall t \in [t_0 - r, t_0+\beta)$,

we have, for $\forall s \in [t_0+\beta_1, t_0+\beta)$, $x_s \in F$ $(\because x_s(u) = x(s+u)$ for $u \in [-r, 0])$

, which implies $\|f(s, x_s)\|^2 \le m(s)$ and $\|g(s, x_s)\|^2 \le m(s)$ (a.s.) by (**).

Hence we have $\mathbb{E}\|f(s, x_s)\|^2 \le m(s)$ and $\mathbb{E}\|g(s,x_s)\|^2 \le m(s)$ (a.s).

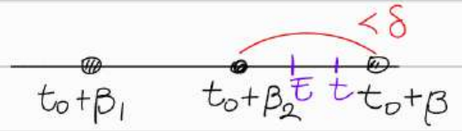
So we get
$$\begin{cases} \int_{\bar{t}}^{t} \mathbb{E}\|f(s,x_s)\|^2 ds \le \int_{\bar{t}}^{t} m(s)\,ds = M(t) - M(\bar{t}) \\ \int_{\bar{t}}^{t} \mathbb{E}\|g(s,x_s)\|^2 ds \le \int_{\bar{t}}^{t} m(s)\,ds = M(t) - M(\bar{t}). \end{cases}$$

pf of Thm 3.1) Since M is absolutely continuous on $[t_0+\beta_1, t_0+\beta)$,

for any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\begin{cases} \cdot\ |t - \bar{t}| < \delta \\ \cdot\ t, \bar{t} \in [t_0+\beta_1, t_0+\beta) \end{cases} \Rightarrow |M(t) - M(\bar{t})| \le \frac{\varepsilon}{2(\beta+1)}$$

Take $\beta_2 > \beta_1$ so that $\beta - \delta < \beta_2 < \beta$



Let $\eta > 0$ be fixed.

By Tchebychev's ineq., we have for $\forall t, \bar{t} \in [t_0+\beta_2, t_0+\beta)$,

$$\mathbb{P}\{\|x(t) - x(\bar{t})\| > \eta\} \le \frac{\mathbb{E}[\|x(t) - x(\bar{t})\|^2]}{\eta^2}$$

$$\le \frac{2(\beta+1)\cdot|M(t) - M(\bar{t})|}{\eta^2} \le \frac{\varepsilon}{\eta^2}$$

By Cauchy criterion, $\exists \lim_{t \to (t_0+\beta)^-} x(t) =: \xi$ with probability 1 and $\xi \in F$.

$(\because x(t) \in F$: compact & ξ is a limit point of $\{x(t)\})$

That is, the solution can be continued by defining $x(t_0+\beta) := \xi$,

but x is noncontinuable. ↯

* (Continuous Extension Theorem)

A ftn $f : (a,b) \to \mathbb{R}^n$ is uniformly continuous

<u>iff</u> it can be defined at the endpoints a and b s.t.

the extended ftn $\tilde{f}(t) := \begin{cases} f(t), & \text{if } t \in (a,b) \\ f(a^+), & \text{if } t = a \\ f(b^-), & \text{if } t = b \end{cases}$

* the seq. $\{X_K(w)\}_{K \geq 1}$ is said to be converge to $X(w)$ "with probability one (w.p.1) or almost surely (a.s.)" if $\mathbb{P}\{w \mid \lim_{K \to \infty} X_K(w) = X(w)\} = 1$.

§3.3. Global Existence.

Thm 3.4 Let $J=\mathbb{R}_+$, $D=\mathbb{R}^n$ be an open set containing $\phi(0)$.

If ① $f\in \mathcal{L}_{ad}(\Omega; L[t_0,t_0+\alpha])$, $g\in \mathcal{L}_{ad}(\Omega; L^2[t_0,t_0+\alpha])$

, where $\alpha>0$ and $[t_0,t_0+\alpha]\subset J$,

are continuous in their second argument ψ.

② $\exists$ two measurable fcts h_1,h_2 (or $h_1,h_2\in PC(\mathbb{R}_+;\mathbb{R}_+)$)

and a continuous increasing convex ftn $K:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ s.t.

(*) $\|f(t,\psi)\|^2\vee\|g(t,\psi)\|^2\le h_1^2(t)+h_2^2(t)K(\|\psi\|_r^2)$ for $\forall(t,\psi)\in\mathbb{R}_+\times\mathcal{L}^2_{\mathcal{F}_t}([-r,0];\mathbb{R}^n)$

i.e. ψ is an $\mathcal{F}_t$-adapted and $\mathbb{E}[\|\psi\|_r^2]<\infty$.

Then, for each $(t,\phi)\in\mathbb{R}_+\times\mathcal{L}^2_{\mathcal{F}_0}([-r,0];\mathbb{R}^n)$, $\exists$ a local $\mathcal{F}_t$-adapted Solution $x=x(t;t_0,\phi(0))$ for (3.1) that can be continued to $[t_0-r,\infty)$

pf) For all $(t,\phi)\in\mathbb{R}_+\times\mathcal{L}^2_{\mathcal{F}_0}([-r,0];\mathbb{R}^n)$, let $x(t)=x(t;t_0,\phi(0))$ be a local solution of (3.1) that is guaranteed by Thm 3.2.

If x is continuable then we know it can be extended to a maximal interval of existence and so assume, WLOG, that x is noncontinuable and let $[t_0-r,t_0+\beta)$ be its maximal interval of existence.

Let $a=\mathbb{E}[\|\phi(0)\|^2]+\mathbb{E}[(\sum_{\{k:t_k\in(t_0,t]\}}\|I(t_k,x_{t_k^-})\|)^2]$, $b=(\beta+1)\beta\bar{h}^2$,

where $\bar{h}=\sup\{h_1(t)\mid\forall t\in[t_0,t_0+\beta]\}$ and $c=\mathbb{E}[\|\phi\|_r^2]$.

Then, $\forall t\in(t_0,t_0+\beta)$,

$$\mathbb{E}[\|x(t)\|^2]\le 4\Big\{\mathbb{E}[\|\phi(0)\|^2]+\mathbb{E}[(\sum_{\{k;t_k\in(t_0,t)\}}\|I(t_k,x_{t_k^-})\|)^2]$$
$$+\beta\int_{t_0}^t\mathbb{E}\|f(s,x_s)\|^2ds+\int_{t_0}^t\mathbb{E}\|g(s,x_s)\|^2ds\Big\}$$

($\because$ ◦ $x(t)=\phi(0)+\int_{t_0}^t f(s,x_s)ds+\int_{t_0}^t g(s,x_s)dW(s)+\sum_{\{k:t_k\in(t_0,t)\}}I(t_k,x_{t_k^-})$

◦ $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2\le4(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)$ if $\alpha_i\in\mathbb{R}$

◦ By Cauchy-Schwartz ineq, $\mathbb{E}\|\int_{t_0}^t f(s,x_s)ds\|^2\le|t-t_0|\int_{t_0}^t\mathbb{E}\|f(s,x_s)\|^2ds$

$t\in(t_0,t_0+\beta)\to|t-t_0|\le\beta$ $\quad\le\beta\int_{t_0}^t\mathbb{E}\|f(s,x_s)\|^2ds$

◦ By Ito Isometry, $\mathbb{E}\|\int_{t_0}^t g(s,x_s)dW(s)\|^2=\int_{t_0}^t\mathbb{E}\|g(s,x_s)\|^2ds$)

$$\le4\Big\{a+b+(\beta+1)\int_{t_0}^t h_2^2(s)K(\mathbb{E}[\|x_s\|_r^2])ds\Big\}$$

($\because$ ◦ By (*), $\beta\int_{t_0}^t\mathbb{E}\|f(s,x_s)\|^2ds+\int_{t_0}^t\mathbb{E}\|g(s,x_s)\|^2ds$

$\le(\beta+1)\int_{t_0}^t \underline{h_1^2(t)}+h_2^2(t)K(\|x_s\|_r^2)ds$

◦ $(\beta+1)\int_{t_0}^t h_1^2(t)dt\le(\beta+1)\int_{t_0}^t\bar{h}^2dt=(\beta+1)\bar{h}^2\underbrace{(t-t_0)}_{\le\beta}\le(\beta+1)\beta\bar{h}^2=:b$)

$\forall t \in [t_0 - r, t_0 + \beta)$
에서 $\mathbb{E}[\|x(t)\|^2]$
$\leq ① + ②$

which implies that

$$\mathbb{E}[\|x_t\|_r^2] \leq \underbrace{C + 4(a+b)}_{①}^{:=B} + \underbrace{4(\beta+1)\int_{t_0}^{t} h_2^2(s)\, K(\mathbb{E}[\|x_s\|_r^2])\, ds}_{②}$$

$$= B + 4(\beta+1)\int_{t_0}^{t} h_2^2(s)\, K(\mathbb{E}[\|x_s\|_r^2])\, ds. \quad \text{for } \forall t \in [t_0, t_0+\beta)$$

Using Bihari's Lemma yields

$$\begin{cases} \mathbb{E}[\|x_t\|_r^2] \leq G^{-1}\left(G(B) + 4(\beta+1)\int_{t_0}^{t} h_2^2(s)\, ds\right) \\ G(B) + 4(\beta+1)\int_{t_0}^{t} h_2^2(s)\, ds \in \mathrm{Dom}(G^{-1}) \end{cases} \quad \text{for } \forall t \in [t_0, t_0+\beta)$$

Bihari Lemma "closed interval"
$0 < \forall \beta_1 < \beta$ & $\beta_1 \to \beta$.

where $G(u) = \int_{0^+}^{u} \frac{ds}{K(s)}$, $u > 0$.

If we set $\tilde{B} := G^{-1}\left(G(B) + 4(\beta+1)\int_{t_0}^{t_0+\beta} h_2^2(s)\, ds\right)$,

then $\mathbb{E}[\|x(t)\|_r^2] \leq \tilde{B} < \infty$ for all $t \in [t_0, t_0+\beta)$

which in turn implies that $\mathbb{E}[\|x(t)\|^2] \leq \tilde{B} < \infty$ for all $t \in [t_0 - r, t_0 + \beta)$

This contradicts with that x is noncontinuable.

§3.4 Uniqueness of Solution

Thm 3.5 Suppose that the assumptions of Thm 3.4 hold and that the functionals

⊗ $f(t,\psi)$, $g(t,\psi)$ are locally Lipschitz in ψ for all $t \in J$.

Then, system (3.1) has a unique solution defined on $[t_0 - r, t_0+\beta)$,

where $0 < \beta \le \infty$ and $[t_0, t_0+\beta) \subset J = \mathbb{R}_+$.

⊗ : For each compact set $F \subset D$, $\exists$ a fct $L(t) \in L^2([t_0, t_0+\alpha] ; \mathbb{R}_+)$

s.t. $\|f(t,\psi_1) - f(t,\psi_2)\| \vee \|g(t,\psi_1) - g(t,\psi_2)\| \le L(t) \cdot \|\psi_1 - \psi_2\|_r$

for all $t \in [t_0, t_0+\alpha]$ defined as in Thm 3.4.

pf) Assume $x(t; t_0, \phi(0))$ and $y(t; t_0, \phi(0))$ are two solutions of (3.1)

on $[t_0, t_0+\beta) \subset J$ $(\beta > 0)$ and assume $x \not\equiv y$ (a.s.).

Since $x(t) = y(t) = \phi(t-t_0)$ for all $t \in [t_0 - r, t_0]$, $\exists\, t \in (t_0, t_0+\beta)$

s.t. $x(t) \neq y(t)$ (a.s.).

Define the stopping time $t_1 := \inf\{t \in (t_0, t_0+\beta) \mid x(t) \neq y(t) \text{ (a.s.)}\}$.

Since $x(t) = y(t)$ for $\forall t \in (t_0, t_1)$, it follows that $x(t_1) = y(t_1)$ (a.s.)

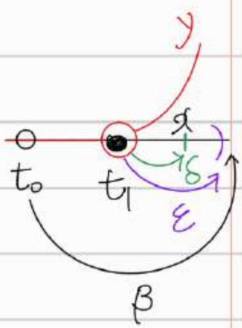
for all possible cases.

⟨case 1⟩ If $t_1 \neq \tau_k(x(t_1^-))$ for $\forall k \in \mathbb{N}$, then

$$x(t_1) = x(t_1^-) = y(t_1^-) = y(t_1). \quad \text{(a.s.)}$$

⟨case 2⟩ If $t_1 = \tau_k(x(t_1^-))$ for some $k \in \mathbb{N}$, then

$$x(t_1) = x(t_1^-) + I(t_1, x_{t_1^-}) = y(t_1^-) + I(t_1, y_{t_1^-}) = y(t_1) \quad \text{(a.s.)}$$



Let $\varepsilon > 0$ be sufficiently small s.t. $t_1 + \varepsilon < t_0 + \beta$ and $\tau_k \notin (t_1, t_1+\varepsilon)$

for all $k \in \mathbb{N}$ and let $S = \{x(t), y(t) \in \mathbb{R}^n \mid t \in [t_0 - r, t_1+\varepsilon]\}$ and $F = \bar{S}$,

then F is clearly closed and bounded subset of $\mathbb{R}^n$ (hence cpt).

and is properly contained in D. ($\because$ by def. $x([t_0-r, t_0+\alpha]) \subset D$, $y([t_0-r, t_0+\alpha]) \subset D$)

Let $\delta > 0$ be sufficiently small number s.t.
- $0 < \delta < \varepsilon$ and
- $2(\delta+1)\int_{t_1}^{t_1+\delta} L(s)^2 ds \le \frac{1}{2}$.

$\rightarrow 0$ as $\delta \to 0$.

Then, for all $t \in [t_1, t_1+\delta]$, $x_t, y_t \in PC([-r, 0] : F)$ and from (2.53)

$\because x(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s) ds + \int_{t_0}^{t} g(s, x_s) dW(s)$ ($t_0 \leftarrow t_1$, $< \varepsilon$: (impulse ×))

$\Rightarrow \mathbb{E}\|x(t) - y(t)\|^2 = \mathbb{E}\left\| \int_{t_1}^{t} (f(s,x_s) - f(s,y_s)) ds + \int_{t_1}^{t} (g(s,x_s) - g(s,y_s)) dW(s) \right\|^2$

$(a+b)^2 \le 2(a^2+b^2)$

$\le 2\left\{ \mathbb{E}\left\| \int_{t_1}^{t} (f(s,x_s) - f(s,y_s)) ds \right\|^2 + \mathbb{E}\left\| \int_{t_1}^{t} (g(s,x_s) - g(s,y_s)) dW(s) \right\|^2 \right.$

Cauch-Schwartz Ineq. Itô isometry

$$\le 2\left\{\delta\cdot\int_{t_1}^{t}\mathbb{E}\left\|\int_{t_1}^{t}(f(s,x_s)-f(s,y_s))\right\|^2 ds+\int_{t_1}^{t}\mathbb{E}\|g(s,x_s)-g(s,y_s)\|^2 ds\right.$$

$$\overset{*}{\le} 2(\delta+1)\int_{t_1}^{t}L(s)^2\,\mathbb{E}\|x_s-y_s\|^2 ds$$

$$\le 2(\delta+1)\int_{t_1}^{t}L(s)^2\cdot\underbrace{\sup_{u\in[t_1,t_1+\delta]}\mathbb{E}\|x(u)-y(u)\|^2}_{\text{independent of } s} ds$$

$$\le \underbrace{2(\delta+1)\cdot\left(\int_{t_1}^{t_1+\delta}L(s)^2 ds\right)}_{\le\frac{1}{2}}\cdot\left(\sup_{u\in[t_1,t_1+\delta]}\mathbb{E}\|x(u)-y(u)\|^2\right)$$

$$\le \frac{1}{2}\cdot\sup_{u\in[t_1,t_1+\delta]}\|x(u)-y(u)\|^2.$$

Since the last inequality is satisfied for all $t\in[t_1,t_1+\delta]$, then this means $\sup_{t\in[t_1,t_1+\delta]}\mathbb{E}\|x(t)-y(t)\|^2=0$.

Since x and y are continuous ftn for all $t\in[t_1,t_1+\delta]$, then $\mathbb{P}\left(\sup_{t\in[t_1,t_1+\delta]}\|x(t)-y(t)\|>0\right)=0$,

$\therefore x\equiv y$ (a.s.).

which implies that $x(t)=y(t)$ (a.s.) for all $t\in[t_1,t_1+\delta]$.

But this contradicts with that $x\not\equiv y$ (a.s.).

Thus, it must be true that (3.1) has a unique solution.